Midterm Exam Solutions

Problem 1

Let $X \sim \text{Expo}(\lambda)$.

(a)

The expectation and variance of X are:

$$E[X] = \int_0^\infty x\lambda e^{-\lambda x}\, dx = \frac{1}{\lambda},$$

$$E[X^2] = \int_0^\infty x^2\lambda e^{-\lambda x}\, dx = \frac{2}{\lambda^2},$$

$$\text{Var}[X] = E[X^2] - (E[X])^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

(b)

The moment generating function $M_X(t)$ is:

$$M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx}\lambda e^{-\lambda x}\, dx = \frac{\lambda}{\lambda - t}, \quad \text{for } t < \lambda.$$

(c)

The derivatives of $M_X(t)$ at $t = 0$ are:

$$\frac{d}{dt} M_X(t)\bigg|_{t=0} = \frac{\lambda}{(\lambda - t)^2}\bigg|_{t=0} = \frac{1}{\lambda} = E[X],$$

$$\frac{d^2}{dt^2} M_X(t)\bigg|_{t=0} = \frac{2\lambda}{(\lambda - t)^3}\bigg|_{t=0} = \frac{2}{\lambda^2} = E[X^2].$$

Problem 2

Let X, Y be two continuous random variables with joint density $\rho_{X,Y}(x, y)$ given by

$$\rho_{X,Y}(x, y) = \frac{12}{y} e^{-3xy^4} \quad \text{for } x > 0,\ y > 1,$$

and 0 otherwise. The marginal density functions are denoted by $\rho_X(x)$ and $\rho_Y(y)$.

(a)

The marginal probability density function $\rho_Y(y)$ is:

$$\rho_Y(y) = \int_0^\infty \rho_{X,Y}(x,y)\,dx = \int_0^\infty \frac{12}{y} e^{-3xy^4}\,dx.$$

Let $u = 3xy^4$, then $du = 3y^4 dx$ and $dx = \frac{du}{3y^4}$. Substituting, we get:

$$\rho_Y(y) = \frac{12}{y}\int_0^\infty e^{-u} \cdot \frac{du}{3y^4} = \frac{12}{y} \cdot \frac{1}{3y^4}\int_0^\infty e^{-u}\,du = \frac{4}{y^5}.$$

Thus,

$$\rho_Y(y) = \frac{4}{y^5}, \quad \text{for } y > 1.$$

(b)

The conditional expectation $E[X \mid Y = 1]$ is:

$$E[X \mid Y = 1] = \int_0^\infty x \cdot \rho_{X|Y}(x \mid 1)\,dx.$$

First, compute the conditional density $\rho_{X|Y}(x \mid 1)$:

$$\rho_{X|Y}(x \mid 1) = \frac{\rho_{X,Y}(x,1)}{\rho_Y(1)} = \frac{\frac{12}{1} e^{-3x\cdot 1^4}}{\frac{4}{1^5}} = 3e^{-3x}.$$

Thus,

$$E[X \mid Y = 1] = \int_0^\infty x \cdot 3e^{-3x}\,dx = \frac{1}{3}.$$

(c)

The conditional expectation $E[X^2 \mid Y = y]$ is:

$$E[X^2 \mid Y = y] = \int_0^\infty x^2 \cdot \rho_{X|Y}(x \mid y)\,dx.$$

First, compute the conditional density $\rho_{X|Y}(x \mid y)$:

$$\rho_{X|Y}(x \mid y) = \frac{\rho_{X,Y}(x,y)}{\rho_Y(y)} = \frac{\frac{12}{y} e^{-3xy^4}}{\frac{4}{y^5}} = 3y^4 e^{-3xy^4}.$$

Thus,

$$E[X^2 \mid Y = y] = \int_0^\infty x^2 \cdot 3y^4 e^{-3xy^4}\,dx.$$

Let $u = 3xy^4$, then $du = 3y^4 dx$ and $dx = \frac{du}{3y^4}$. Substituting, we get:

$$E[X^2 \mid Y = y] = \int_0^\infty \left(\frac{u}{3y^4}\right)^2 \cdot 3y^4 e^{-u} \cdot \frac{du}{3y^4} = \frac{1}{9y^8}\int_0^\infty u^2 e^{-u}\,du = \frac{2}{9y^8}.$$

Problem 3

Let $X_0 = 3$, $X_n = \sum_{k=1}^{n} \xi_k$, where $\{\xi_k\}_{k\geq 1}$ is a sequence of independent and identically distributed random variables such that $P(\xi_k = 1) = \frac{2}{3}$ and $P(\xi_k = -1) = \frac{1}{3}$. Define

$$\tau = \min\{n > 0 : X_n = 0 \text{ or } X_n = 6\}.$$

(a)

The process $\left(\frac{1}{2}\right)^{X_n}$ is a martingale because:

$$E\left[\left(\frac{1}{2}\right)^{X_{n+1}} \mid \mathcal{F}_n\right] = \left(\frac{1}{2}\right)^{X_n} \cdot \left(\frac{2}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot 2\right) = \left(\frac{1}{2}\right)^{X_n}.$$

(b)

Take $B = [6, \infty) \cup (-\infty, 0]$ in Lemma 2.15 of the lecture notes..

(c)

We need to prove that $\mathbb{P}[\tau < +\infty] = 1$ and deduce that $\mathbb{P}[X_\tau \in \{0, 6\}] = 1$.

Define the event A_k as the event that the sequence $\{\xi_{k+1}, \xi_{k+2}, \ldots, \xi_{k+5}\}$ consists of all $+1$ or all -1. That is:

$$A_k = \{\xi_{k+1} = \xi_{k+2} = \cdots = \xi_{k+5} = +1\} \cup \{\xi_{k+1} = \xi_{k+2} = \cdots = \xi_{k+5} = -1\}.$$

Since ξ_k are i.i.d. with $P(\xi_k = 1) = \frac{2}{3}$ and $P(\xi_k = -1) = \frac{1}{3}$, we have:

$$P(A_k) = \left(\frac{2}{3}\right)^5 + \left(\frac{1}{3}\right)^5 = \frac{32}{243} + \frac{1}{243} = \frac{33}{243} = \frac{11}{81}.$$

If any A_k occurs, then within the next 5 steps, X_n will either increase or decrease by 5. Since $X_0 = 3$, if X_n increases by 5, it will reach at least 6; if it decreases by 5, it will reach at most 0. Therefore, if any A_k occurs, then $\tau \leq k + 5$.

$\mathbb{P}[\tau > 5n]$ equals to the Probability that A_k not occurs for k $\in$ (0,5n) is smaller than $(1 - P(A_k))^n$, which tends to 0 as n tends to infinity.

Thus we conclude that $\mathbb{P}[\tau < +\infty] = 1$

Since $\tau < +\infty$ almost surely, and X_n reaches either 0 or 6 at time τ, we have:

$$\mathbb{P}[X_\tau \in \{0, 6\}] = 1.$$

(d)

Using the optional stopping theorem:

$$E\left[\left(\frac{1}{2}\right)^{X_\tau}\right] = \left(\frac{1}{2}\right)^0 \cdot \mathbb{P}[X_\tau = 0] + \left(\frac{1}{2}\right)^6 \cdot \mathbb{P}[X_\tau = 6].$$

Since $\left(\frac{1}{2}\right)^{X_n}$ is a martingale, we have:

$$E\left[\left(\frac{1}{2}\right)^{X_\tau}\right] = \left(\frac{1}{2}\right)^{X_0} = \left(\frac{1}{2}\right)^3 = \frac{1}{8}.$$

Let $p = \mathbb{P}[X_\tau = 6]$, then:

$$\frac{1}{8} = 1 \cdot (1-p) + \frac{1}{64} \cdot p.$$

Solving for p, we get:

$$\frac{1}{8} = 1 - p + \frac{p}{64}, \quad \Rightarrow \quad \frac{1}{8} = 1 - \frac{63p}{64}.$$

Thus,

$$p = \frac{64}{63} \cdot \left(1 - \frac{1}{8}\right) = \frac{64}{63} \cdot \frac{7}{8} = \frac{8}{9}.$$

Therefore,

$$\mathbb{P}[X_\tau = 6] = \frac{8}{9}.$$

(e)

The process $X_n - \frac{1}{3}n$ is a martingale because:

$$E\left[X_{n+1} - \frac{1}{3}(n+1) \mid \mathcal{F}_n\right] = X_n + E[\xi_{n+1}] - \frac{1}{3}(n+1) = X_n - \frac{1}{3}n.$$

(f)

Using the optional stopping theorem:

$$E[X_\tau - \frac{1}{3}\tau] = E[X_0] = 3.$$

Since $X_\tau \in \{0, 6\}$, we have:

$$E[X_\tau] = 0 \cdot \mathbb{P}[X_\tau = 0] + 6 \cdot \mathbb{P}[X_\tau = 6] = 6 \cdot \frac{8}{9} = \frac{16}{3}.$$

Thus,

$$E[\tau] = 3\left(E[X_\tau] - 3\right) = 3\left(\frac{16}{3} - 3\right) = 3 \cdot \frac{7}{3} = 7.$$

Problem 4

See Question 1 in Homework 5.